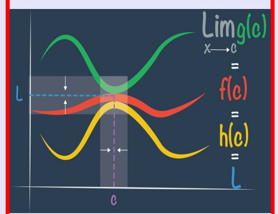
**Math 261**

**Fall 2022**

**Lecture 51**




Feb 19-8:47 AM

Find $f_{ave}$ for $f(x) = \sec^2 x$ on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx \quad \text{if } f(x) \text{ is cont. on } [a,b]$$

$f(x) = \sec^2 x$ is cont. on $\left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$

$$f_{ave} = \frac{1}{\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)} \int_{-\pi/4}^{\pi/4} \sec^2 x\,dx = \frac{1}{\frac{\pi}{2}} \cdot 2 \cdot \int_0^{\pi/4} \sec^2 x\,dx$$

$$= \frac{4}{\pi} \cdot \tan x \Big|_0^{\pi/4} = \frac{4}{\pi}\left[\underbrace{\tan\frac{\pi}{4}}_{1} - \underbrace{\tan 0}_{0}\right] = \boxed{\frac{4}{\pi}}$$

Nov 30-8:46 AM

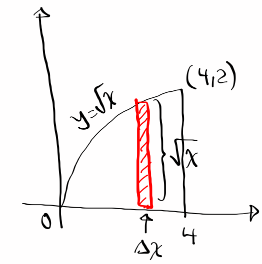
The base of Solid S is the region enclosed by $y=\sqrt{x}$, $y=0$, and $x=4$.

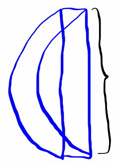
Cross-Sections $\perp$ x-axis.

Cross-sections are Semicircles.

Diameters across the base.

Find its volume.



Diameter of the circle

$d=\sqrt{x}$

$r=\frac{\sqrt{x}}{2}$

Area of Circle

$\pi r^2$

we want Semicircle

$\frac{\pi r^2}{2}$

$$\text{Volume} = \frac{\pi r^2}{2}\cdot \Delta x = \frac{\pi}{2}\left(\frac{\sqrt{x}}{2}\right)^2\cdot \Delta x = \frac{\pi}{8}\cdot x\cdot \Delta x$$

$$V=\int_0^4 \frac{\pi}{8}\cdot x\,dx = \frac{\pi}{8}\cdot\frac{x^2}{2}\Big|_0^4 = \frac{\pi}{16}(4^2-0^2) = \boxed{\pi}$$

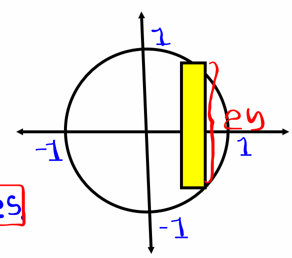
Nov 30-8:51 AM

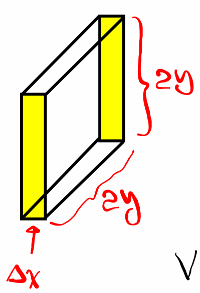
Consider a Solid S whose base is enclosed region by $x^2+y^2=1$.

Cross-Sections $\perp$ x-axis.

Cross-Sections are Squares

and one Side is in the base.




$$\text{Volume} = 2y\cdot 2y\cdot \Delta x$$

$$= 4y^2\cdot \Delta x$$

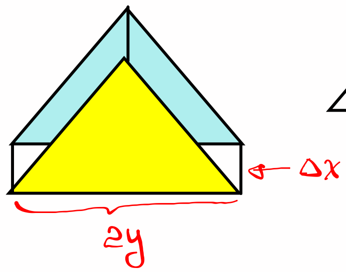
$$= 4(1-x^2)\cdot \Delta x$$

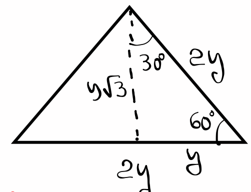
$$V=\int_{-1}^{1} 4(1-x^2)\,dx = 4\cdot 2\int_0^1 (1-x^2)\,dx$$

$$= 8\left(x-\frac{x^3}{3}\right)\Big|_0^1 = 8\cdot\frac{2}{3} = \frac{16}{3}$$

Nov 30-8:58 AM

Repeat the last problem if cross-sections are equilateral triangles.




$$\text{Area} = \frac{b \cdot h}{2} = \frac{2y \cdot y\sqrt{3}}{2} = y^2\sqrt{3}$$

$$V = \text{Area of triangle} \cdot \Delta x = y^2\sqrt{3} \cdot \Delta x = \sqrt{3}(1-x^2)\Delta x$$

$$V = \int_{-1}^{1} \sqrt{3}(1-x^2)\,dx = \sqrt{3} \cdot 2\int_0^1 (1-x^2)\,dx \qquad V = \boxed{\frac{4\sqrt{3}}{3}}$$

$$= 2\sqrt{3}\left[x - \frac{x^3}{3}\right]\Big|_0^1 = 2\sqrt{3} \cdot \frac{2}{3}$$

Nov 30-9:06 AM

Find $f_{ave}$ for $f(x) = \frac{x}{(5x^2+1)^2}$ on $[0,1]$

$f(x) = \frac{x}{(5x^2+1)^2}$ is cont. everywhere.

$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$ if $f(x)$ is cont. on $[a,b]$

$$f_{ave} = \frac{1}{1-0}\int_0^1 \frac{x}{(5x^2+1)^2}\,dx$$

$u = 5x^2+1$

$du = 10x\,dx$

$\frac{du}{10} = x\,dx$

$x=0 \to u=1$

$x=1 \to u=6$

$$= \int_1^6 \frac{1}{u^2} \cdot \frac{du}{10}$$

$$= \frac{1}{10}\int_1^6 u^{-2}\,du = \frac{1}{10} \cdot \frac{u^{-1}}{-1}\Big|_1^6$$

$$= \frac{-1}{10}\left[\frac{1}{u}\right]\Big|_1^6 = \frac{-1}{10}\left(\frac{1}{6} - 1\right) = \frac{-1}{10} \cdot \frac{-5}{6} = \boxed{\frac{1}{12}}$$

Nov 30-9:13 AM

Evaluate $\int_1^2 \frac{dx}{x^2-6x+9}$

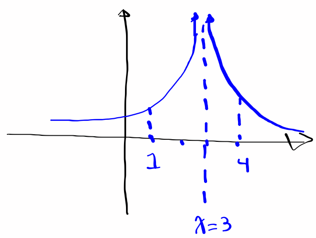
$\int_1^2 \frac{dx}{(x-3)^2}$ $\quad u = x-3$, $du = dx$ $\quad x=1 \to u=-2$, $x=2 \to u=-1$

$$= \int_{-2}^{-1} \frac{1}{u^2}\,du = \int_{-2}^{-1} u^{-2}\,du = \frac{u^{-1}}{-1}\Big|_{-2}^{-1} = -\left(\frac{1}{u}\Big|_{-2}^{-1}\right)$$

$$= -\left[\frac{1}{-1} - \frac{1}{-2}\right] = 1 - \frac{1}{2} = \boxed{\frac{1}{2}}$$

Redo for [1,4]

$\int_1^4 \frac{dx}{x^2-6x+9} = \int_1^4 \frac{dx}{(x-3)^2}$ $\quad$ Calc. II $f(x)$ is not cont. on [1,4]




Requires limits.

Calc. II.

Nov 30-9:19 AM

find $\int_{-1}^{4} \frac{x}{\sqrt{x+5}}\,dx \to$ Cont on $(-5, \infty)$

$u = \sqrt{x+5}$

$u^2 = x+5$

$u^2 - 5 = x$ $\quad x=-1 \to u=2$

$2u\,du = dx$ $\quad x=4 \to u=3$

$$\int_2^3 \frac{u^2-5}{\cancel{u}} \cdot 2\cancel{u}\,du = 2\int_2^3 (u^2-5)\,du = 2\left[\frac{u^3}{3} - 5u\right]\Big|_2^3$$

$$= 2\left[\left(\frac{27}{3} - 15\right) - \left(\frac{8}{3} - 10\right)\right] = 2\left[-6 - \frac{8}{3} + 10\right] = 2\left(4 - \frac{8}{3}\right)$$

$$= 2 \cdot \frac{4}{3} = \boxed{8/3}$$

Nov 30-9:29 AM

Find $\int_{-2}^{0} x\, f(x^2)\, dx$ if $\int_{0}^{4} f(x)\,dx = 1$.

$u = x^2$ $\quad$ $x=-2 \to u=4$

$du = 2x\,dx$ $\quad$ $x=0 \to u=0$

$\frac{du}{2} = x\,dx$

$$\int_{-2}^{0} x\, f(x^2)\,dx = \int_{4}^{0} f(u)\cdot \frac{du}{2}$$

$$= -\frac{1}{2}\int_{0}^{4} f(u)\,du = -\frac{1}{2}\cdot \boxed{1} = \boxed{-\frac{1}{2}}$$

Nov 30-9:34 AM

For positive integer $n$,

Find $\int_{0}^{1} x(1-x)^n\,dx$

$u = 1-x \to x = 1-u$

$du = -dx$ $\quad$ $x=0 \to u=1$

$-du = dx$ $\quad$ $x=1 \to u=0$

$$\int_{0}^{1} x(1-x)^n\,dx = \int_{1}^{0} (1-u)u^n \cdot -du = \int_{0}^{1} (1-u)\cdot u^n\,du$$

$$= \int_{0}^{1} (u^n - u^{n+1})\,du = \left(\frac{u^{n+1}}{n+1} - \frac{u^{n+2}}{n+2}\right)\Bigg|_{0}^{1}$$

$$= \frac{1}{n+1} - \frac{1}{n+2} = \frac{n+2-(n+1)}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)}$$

Nov 30-9:41 AM

Find $\int_{-\pi/4}^{\pi/4} \tan^4 x\, dx = 2\int_0^{\pi/4} \tan^2 x \cdot \tan^2 x\, dx$

$1 + \tan^2 x = \sec^2 x$

$\tan^2 x = \sec^2 x - 1$

$= 2\int_0^{\pi/4} \tan^2 x(\sec^2 x - 1)\, dx$

$= 2\int_0^{\pi/4} (\tan^2 x \sec^2 x - \tan^2 x)\, dx = 2\left(\frac{1}{3} - 1 + \frac{\pi}{4}\right)$

$= \boxed{\phantom{xx}}$

$\int \tan^2 x \sec^2 x\, dx =$

$u = \tan x$

$du = \sec^2 x\, dx$

$x = 0 \quad u = 0$

$x = \frac{\pi}{4} \quad u = 1$

$\int_0^1 u^2\, du = \left.\frac{u^3}{3}\right|_0^1 = \boxed{\frac{1}{3}}$

$\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx = \left.(\tan x - x)\right|_0^{\pi/4}$

$= 1 - \frac{\pi}{4}$